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On the continuity equation

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Abstract
In this paper, expediency of modification of the continuity equation is discussed. An example of a problem which has no solution within the framework of the standard approach is presented. The origin of the contradiction is discovered and means to overcome it are considered. New derivation of the so-called mass density diffusion equation is suggested. The averaging procedure and correct averaging of the model equations (the continuity equation including) are described.

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1. Introduction

From time to time, papers appear which discuss or use modifications of the continuity equation via redefinition of the mass flux density. The authors of [1, 2] were the first who considered a generalization of this kind. They suggested adding terms proportional to gradients of the mass density and temperature to the standard mass flux density expression. These papers caused critical reports [3, 4], which declared the above-mentioned considerations to be false and maintained that the mass transfer, irrespective of its causes, is fully characterized by the quantity $\rho\vec{v}$, where ρ is the mass density and $\vec{v}$ is a macroscopic velocity of the media. The problem itself was called objectless. Nevertheless, later the same problem was considered in [5–7] from the standpoint of kinetic theory. Along with self-diffusion, an additional input to the mass flux due to particle mobility in the field of external forces was taken into account.

In addition, a number of papers exist which deal with a new approach in computational fluid dynamics based on the substitution of the standard Navier–Stokes system for the so-called quasihydro- or quasigasdynamic equations (see, e.g., [8, 9]). The derivational procedure of such equations includes redefinition of the mass flux density and leads to a non-standard continuity equation. The new system of equations differs from the standard one by small dissipative terms which increase the effectiveness of the finite-difference algorithms.

All the above-cited articles consider modified continuity equations as substitutions for the standard one. Meanwhile, there is a series of papers devoted to the computation of sea currents which also exploit the changed continuity equation although in a different way. In this case, the system of equations of the model includes not only a new variant of the continuity equation but the standard one as well. Beginning from [10], such an equation is derived from the heat equation, diffusion equation for salinity and the linear equation of state. It differs from the known continuity equation by a diffusion term on the right-hand side, which means implicit redefinition of the mass flux density. This equation was called the *mass density diffusion equation* and is used to close the system of equations of the model instead of the heat equation and the equation of diffusion of salinity.

Although all the authors point out the positive effect of substitution of the continuity equation or the whole Navier–Stokes system for new equations, a common viewpoint is not yet developed and none of the new variants of the continuity equation have found widespread acceptance. This justifies another attempt to investigate the problem. Contrary to some researchers which prefer to base their reasonings on the consideration of the extensions of the Boltzmann equation, we shall closely follow the framework of continuum mechanics. Integral laws which were formulated for real fluids are then regarded to hold for the large-scale motions of continuous media. In addition, we shall focus our attention on the continuity equation, since all the above-mentioned modifications of the Navier–Stokes system change it drastically. With this in mind, it is suggested to discuss the following questions.

(1) What are the drawbacks of the standard continuity equation? Is it satisfactory in the sense that the system of the fluid mechanics equations, the continuity equation included, possesses a solution for any reasonably posed (from the physical point of view) problem?

Since the answer for the second question is negative (see a counterexample in section 2), it makes sense to discuss two more questions.

(2) What must the continuity equation take into account and what must it disregard due to the general derivational history of the fluid mechanics equations and the order of introduction (definition) of new quantities (densities of mass, forces, momentum, etc).

(3) How must a ‘correct’ continuity equation look and what is its fitting place in the system of fluid mechanics equations?

The current paper is organized as follows. Section 2 demonstrates a simple example of a problem which has no solution within the standard hydromechanics of perfect or viscous fluid. The corresponding system of equations includes two contradicting statements. They are the continuity equation and the heat equation. Possible modifications of the continuity equation are considered in section 3. Next, section 4 is devoted to analysis of the standard averaging procedure. It also includes a description of a new averaging method suggested in [12]. This method is used further in section 5 and allows the derivation of the mass density diffusion equation (i.e. non-standard continuity equation). Such an equation satisfies the requirements formulated in section 3. As has been said, this equation is often used in geophysical problems (see, e.g., [13, 14]), although it is obtained differently being based on other considerations (see [10]). Critical analysis of these considerations is presented in section 6 together with some concluding remarks.

2. Mass density jump problem

2.1. Counterexample

An example of an experiment which cannot be described using the standard system of fluid mechanics equations can easily be given. It leads to an unsolvable problem and, thus, contradicts observations.

Consider an evolution of a mass density disturbance $\rho(t,z)$ connected with a 1D temperature disturbance $\theta(t,z)$ in a fluid at rest. Let it be a horizontally homogeneous layer with thickness h and have stable (in order to avoid generation of macroscopic motions) vertical stratification made by a temperature jump. The surface pressure (p_s) and surface temperature (θ_s) as well as bottom temperature (θ_b) are considered to be constant. The known evolution of such a system consists of dissipation of initial disturbance and relaxation of initial profile to a linear one.

Let us write out the system of equations of a viscous heat conducting fluid in the gravitational field. This system consists of the continuity equation, the equation of motion (the momentum balance equation), the heat equation (energy balance) and the equation of state (see, e.g., [15]):

$$\begin{aligned} &d_t\rho + \rho\,\mathrm{div}\vec{\overline{v}} = 0,\\ &\rho d_t\vec{\overline{v}} = -\nabla p + 2\mu\,\mathrm{div}\mathsf{D} + \rho\vec{g},\\ &\rho C_p d_t\theta = d_t p + 2\mu\mathsf{D}:\mathsf{D} + \lambda\Delta\theta,\\ &\rho = \rho(p,\theta). \end{aligned} \tag{1}$$

Here ρ is the mass density, $\vec{\overline{v}}$ is the macroscopic velocity, p is the pressure, $\vec{g}$ is the gravity acceleration, μ is the dynamic viscosity coefficient, θ is the temperature, D is the deformation rate tensor and C_p is the specific heat at constant pressure. Four quantities $(\rho, \vec{\overline{v}}, p, \theta)$ are unknowns. Thus, the system of four independent equations is closed.

In the case of horizontally homogeneous continuum with zero macroscopic velocity the system simplifies to

$$\begin{aligned} &\partial_t\rho = 0,\\ &0 = \partial_z p + \rho g,\\ &\rho C_p\partial_t\theta = \partial_t p + \lambda\partial_{zz}\theta,\\ &\rho = \rho(p,\theta), \end{aligned} \tag{2}$$

where z is a downward vertical coordinate and $g = |\vec{g}|$. The corresponding initial and boundary conditions are as follows:

$$\begin{aligned} &\theta|_{t=0} = \theta_0(z),\\ &p|_{t=0} = p_0(z),\\ &\rho|_{t=0} = \rho(p_0,\theta_0),\\ &p|_{z=0} = p_s,\\ &\theta|_{z=0} = \theta_s,\\ &\theta|_{z=h} = \theta_b. \end{aligned} \tag{3}$$

Here $\theta_0(z) = \begin{cases}\theta_s(z), & z\in(0,z_*],\\ \theta_b(z), & z\in[z_*,h],\end{cases}$ and $z_*\in[0,h]$ is a depth where $\theta_0(z)$ is discontinuous.

The system (2) is overdetermined and, obviously, has no solution. Indeed, according to the first equation of (2) the mass density profile conserves in time. In this case, the second equation describes the pressure field with a time independent vertical profile. If the surface pressure is constant, then p depends on the vertical coordinate only. The third equation becomes the simple one-dimensional heat equation $\partial_t\theta = \kappa\partial_{zz}\theta$, $\kappa = \frac{\lambda}{\rho C_p}$, and describes time evolution of the initial temperature profile in accordance with the boundary conditions. In the present case, the evolution means dissipation of temperature jump due to thermal conductivity and its relaxation to a linear temperature profile. The last equation, in turn, allows one to find the corresponding evolution of the initial mass density distribution. Since the existence of such evolution contradicts the first equation, the whole *problem has no solution*.

2.2. Analysis of the contradiction

Why does this contradiction exist? From our viewpoint, it is caused by the incorrect derivation of the model equations and, more precisely, the continuity equation. Mathematically, the cause lies in different types of equations. The continuity equation is of hyperbolic type, whereas the heat equation is of parabolic type. The first one describes transport of an initial condition along characteristics without change. Solutions of the second equation satisfy the maximum principle and describe dissipation of initial disturbance with respect to time.

In the framework of the standard fluid mechanics a supposition that the contradiction comes from the incorrect derivation of equations is a hypothesis since the mass and energy conservation laws, which lead to the continuity and the heat equations, are independent. This hypothesis is verified by further considerations. However, if one considers four-dimensional causal fluid mechanics [16, 17], then both the differential equations follow from two interpretations of one and the same measure conservation law, and the origin of the contradiction becomes evident: one integral conservation law leads to differential equations of different types.

Note that the same result could be achieved using the system of equations of the perfect fluid, which also includes the heat equation. Both fluid models are based on the continuity hypothesis which leads to the parametric method of description of small-scale motions. The corresponding parameters are called the internal energy and the temperature, which is associated with the internal energy density. Internal energy is interpreted as a mean kinetic energy of small-scale movements. Its introduction is actually connected with averaging of the system of differential equations of continuum, which follow from the general integral principles and are formally valid for motions of arbitrary scales.

The initial subject of investigation of continuum mechanics is the motion of a very large totality of material particles. The continuum hypothesis, which is used as an auxiliary assumption, allows substitution of the above-mentioned totality for continuous media and of individual characteristics of moving particles for field functions. Finally, these fields should be considered as a result of averaging of a set of individual characteristics and the whole theory as an approximate one, which describes the travel of the totality of fictitious points along averaged trajectories.

Averaging of trajectories causes substitution of the initial velocity field $\vec{v}$ for the averaged one $\vec{\overline{v}}$ and the kinetic energy T of the points of the media for the kinetic energy of mean motion K, which generally differs from T. This difference is taken into account parametrically via the introduction of the so-called internal energy $E = T - K$. All this is combined in a postulate known as the first law of thermodynamics. Consideration of the energy exchange gives birth to the internal energy balance equation.

As regards the continuity equation, it is not averaged, remains unchanged and implicitly appears to be valid for arbitrary small parts of continuum. In the case of a turbulent fluid averaging of the continuity equation is performed explicitly, although with the same result. The point is that the acceptance of an auxiliary incompressibility hypothesis foregoes averaging (see, e.g., [18]) and this allows identification of continuity of continuum with non-divergence of the velocity field. Averaging of the linear equation $\mathrm{div}\vec{v} = 0$ preserves its form and gives $\mathrm{div}\overline{\vec{v}} = 0$. As has been shown above, such reasoning sometimes leads to unsolvable problems and contradictions with observations.

3. Analysis of derivational history of the continuum mechanics equations

In order to find out what must and what cannot be taken into account by the continuity equation let us consider the general procedure of the derivation of the continuum mechanics equations and the order of introduction (definition) of new quantities. The successive derivation of the model equations consists of the following steps.

(1) Gathering of experimental data and formulation of consistent integral relations concerning real fluid motion. There are three such relations:
   - (a) the mass M conservation law $d_t M = 0$,
   - (b) the law of motion $d_t m = f$, where m is a momentum and f is an external force,
   - (c) the energy conservation law.

(2) Acceptance of the continuity hypothesis, i.e. substitution of totality of material particles for continuum of points, and assuming that the integral relations obtained are valid for continuum. These integral relations are considered as postulates within current theory.

(3) The introduction of densities of corresponding integral quantities and the derivation of differential equations based on integral relations which are formally valid within an arbitrarily small vicinity of every point of continuum. They are
   - (a) the continuity equation,
   - (b) the equation of motion,
   - (c) the energy density balance equation.

(4) In view of the bounds of applicability of the theory, the explicit consideration is left for large-scale processes only. Initial equations are substituted for averaged ones and small-scale processes are described parametrically. The result should be as follows:
   - (a) the averaged continuity equation,
   - (b) the averaged equation of motion,
   - (c) the internal energy density balance equation or the total energy density conservation law.

Let us make two remarks on the above-presented scheme.

First, the postulates, which are at the basis of the theory, must be independent and they are such. In this regard, the discussion of the mass conservation law as well as the derivation of the corresponding differential equation, generally speaking, does not need postulation of the other two laws and may be done independently.

As is known, the derivation of differential equations from integral relations requires the introduction of local characteristics of continuum. The continuity equation needs only two such characteristics: the mass density ρ and the velocity $\vec{v}$ of the points of the medium. At this step of development of the theory other local characteristics are not known, i.e. need not be defined and introduced. Thus, the continuity equation cannot and must not contain terms which describe effects of external forces, temperature or the like. Mass density itself, surely, can depend on forces. However, such a dependence is implicit and is connected with the velocity field. An explicit dependence also can take place, but it is described by another equation, namely the equation of state. Solely the parameters (coefficients) of the differential mass conservation law contain quantities different from ρ and $\vec{v}$ in case these parameters were introduced in the equation.

Second, the widely accepted procedure of the derivation of equations of the model of continuum (see, e.g., [15, 19–25] and others) differs from the above-written in its last step, which is done in part. Actual averaging is applied to the kinetic energy density balance equation only, or together with the equation of motion in the case of a viscous fluid. In addition, in the latter case the averaging is accomplished implicitly via the introduction of the so-called viscous stress tensor. This operation does not affect the continuity equation, and it retains its initial form. The resulting system of equations (1) is, however, self-contradictory as has been shown in the example discussed above.

If, in contrast, the averaging is applied to the continuity equation in accordance with the above-written scheme, then the so-called equation of the mass density diffusion may be obtained. Corresponding modification of the mass density jump problem removes the mentioned contradiction (see section 6). The derivation of this equation is described in what follows.

4. On the averaging procedure

As has been noted, the fluid mechanics being based on the continuity hypothesis is a macroscopic theory which describes the most probable behaviour of a system consisting of a large number of particles. Accordingly, the hydromechanical equations must describe averaged movements of the points of the body along smoothed trajectories.

The usual averaging procedure regards each hydromechanical variable independently as a random process (see, e.g., [18]) and considers it as a sum of a mean (smoothed) component and a random (pulse) one. Such an approach leads, however, to immense expressions with numerous new unknowns and, in pure form, is not used. A series of simplifying assumptions (e.g. Boussinesq approximation) are made instead immediately before averaging and the actual averaging is applied to velocity components only. Thus, the averaging procedure, in practice, is not general and may be used only if the above-mentioned assumptions can be formulated.

However, an alternative view on the averaging problem is possible. First, it should be noted that unlike integral parameters of continuum (e.g. mass, charge, volume, etc), which are potentially measurable, local characteristics (such as densities and velocities defined at a point) are non-measurable in principle[1]. They must be calculated in accordance with their definitions and imposed constraints, i.e. system of equations. This means that initial hydromechanical fields are actually not independent. In contrast, they are closely connected with each other, and their connection is determined by a system of equations, e.g. perfect of viscous fluid equations (see [18] for discussion of applicability of the fluid mechanics equations to turbulent currents). An arbitrary random disturbance in any of these fields gives birth to disturbances in all others. At the same time, these disturbances are such that all disturbed fields still satisfy the above-mentioned system of equations. In other words, the induced disturbances are not random and disappear in the absence of the initial one.

[1] Widely used experimental determination (i.e. measurement) of values of local parameters (velocities, densities, temperature, pressure, etc) should not mislead, since each local parameter is defined to be a characteristic of a point while measurement of such a parameter is the determination of the ratio of two measures of a finite part of a real fluid. Actually, each measurement is an averaging.

Let one of the fields be considered independent and assume it to be a sum of two components, a smooth and a random. In this case, all other fields which are functionally connected with the chosen one will also contain random components. On the other hand, these fields may be regarded as smooth (containing no random components) if they are calculated using averaged values of the independent quantity only. Obviously, such an independent quantity should be one of the primary characteristics of continuum (in the sense of the order of their introduction in the theory). The velocity field of the points of continuum seems to be a reasonable choice.

Indeed, the mass M of the fluid body and its volume $V(t)$ act as initial notions of the hydromechanical theory (see, e.g., [25]). It may easily be shown that the volume change rate $d_t V$ is defined by the velocity vector field $\vec{v}$. Let V_0 be a volume of a fluid body at some reference time value t_0. Then the current volume may be calculated using the formula

$$V(t) = \int_{V_0} J\,\mathrm{d}V,$$

where J is the Jacobian of the Lagrange to Euler coordinates change. The volume change rate is then given as follows:

$$d_t V = \int_{V_0} d_t J\,\mathrm{d}V = \int_V \frac{1}{J} d_t J\,\mathrm{d}V = \int_V \mathrm{div}\vec{v}\,\mathrm{d}V. \tag{4}$$

This gives a differential relation which holds in an arbitrary point of continuum (see e.g., [15])

$$\frac{1}{J} d_t J = \mathrm{div}\vec{v}. \tag{5}$$

It should be noted that an averaging problem, as well as scale separation of motion, may be considered at this step of developing the theory although neither mass density nor force densities have been defined and introduced.

Thus, the only one time-dependent measure which may contain a random component is the volume of the fluid body. Its evolution is governed by the velocity field in accordance with (4). The velocity vector $\vec{v}$ and the Jacobian J are those local characteristics of the fluid which should be considered as primary (in the above-indicated sense) unlike mass density, pressure and so forth. Since the moving continuum is usually described in terms of velocity, it is convenient to regard velocity as a characteristic which undergoes averaging. In addition, this is in agreement with usual practice.

The averaging operator (denoted hereafter by angel brackets $\langle\cdot\rangle$) must be a continuous linear projector mapping a set of non-smoothed objects onto a set of averaged objects [18]. Let us represent each non-averaged trajectory $\lambda(t)$ as a sum $\bar{\lambda}(t) + \lambda'(t)$ of a smooth curve $\bar{\lambda}(t)$ and a pulse curve $\lambda'(t)$. Then the velocity vector $\vec{v}(t)$ may be written as a corresponding sum $\bar{\vec{v}}(t) + \vec{v}'(t)$. The first term is a vector tangent to a smoothed trajectory whereas the second term is tangent to a pulse curve. The vector $\vec{v}$ is understood as a sum of two elements of a tangent vector space.

5. Mass density diffusion equation

5.1. Standard averaging of the continuity equation

One of the main ideas underlying fluid mechanics consists of the scale separation of motion. Large-scale motions are described explicitly, whereas small-scale motions are described parametrically. The equations of a large-scale evolution of the fluid are derived by averaging the source differential conservation laws. Here, only the mass conservation law

$$\partial_t \rho + \mathrm{div}\rho\vec{v} = 0 \tag{6}$$

is required. In the framework of incompressible fluid mechanics (which corresponds to the Navier–Stokes or Reynolds equations of motion) this equation is usually written as a sum of two zero-valued terms

$$-\frac{1}{\rho}d_t\rho = \mathrm{div}\vec{v} = 0.$$

Averaging of these terms gives the continuity equation of mean motion

$$\langle \mathrm{div}\vec{v}\rangle = \mathrm{div}\vec{\bar{v}} = 0. \tag{7}$$

In addition, the equation $\langle \frac{1}{\rho}d_t\rho\rangle = 0$ is treated as

$$\frac{1}{\bar{\rho}}d_t\bar{\rho} = 0, \tag{8}$$

which requires special assumptions (see, e.g., [18]).

5.2. Mass density diffusion equation

Mass density is calculated using the velocity field and its value depends on what we call a velocity. That is why densities which are obtained using equations (6) and (8) are generally different. Note that the quantity $\bar{\rho}$ in (8) must be understood as a density which is calculated using the mean velocity field $\vec{\bar{v}}$, not as a result of the averaging of some independent stochastic quantity ρ. The difference $\rho' \equiv \rho - \bar{\rho}$ may be called a pulse of the mass density. However, it should be taken into account that ρ' is not an independent stochastic quantity, but is induced by small-scale variations of the velocity field (velocity pulsations) and is identical to zero in the case where these variations are absent. Thus, the density ρ corresponds to the initial velocity field $\vec{v}$, whereas the density $\bar{\rho}$ corresponds to the averaged field $\vec{\bar{v}}$.

At the same time, it is clear that small-scale velocity components (which are described implicitly) are responsible for some displacement of mass and its redistribution. Together with the mean current they participate in the arranging of the mass density field. For example, if in the absence of the mean current there exists a stable density stratification the velocity pulsations may cause smoothing of density disturbances (see the detailed discussion of this topic in [12]). In contrast to [3, 4], we think that it makes sense to consider diffusion phenomena not only in the case of mixtures but in the case of single fluids as well. Such a self-diffusion process must be understood as a small scale transport of matter due to velocity pulses.

To take into account these processes, let us derive an equation which connects the density field ρ with the mean velocity field. We write velocity as $\vec{v} = \vec{\bar{v}} + \vec{v}'$, substitute it into equation (6) and average

$$\langle \partial_t\rho + \mathrm{div}\rho(\vec{\bar{v}} + \vec{v}')\rangle = \langle\partial_t\rho\rangle + \mathrm{div}(\langle\rho\vec{\bar{v}}\rangle + \langle\rho\vec{v}'\rangle) = 0. \tag{9}$$

Now there are two ways of reasoning.

(1) One may neglect the third term in (9), assume that the mass density flux is equal to $\langle\rho\vec{\bar{v}}\rangle = \bar{\rho}\,\vec{\bar{v}}$ and obtain an equation which describes mass density evolution due to the mean velocity field:

$$\partial_t\bar{\rho} + \mathrm{div}(\bar{\rho}\vec{\bar{v}}) = 0.$$

This is the standard approach and the standard continuity equation.

(2) It is possible to neglect nothing, consider the mass density flux to be equal to

$$\langle\rho\vec{\bar{v}}\rangle + \langle\rho\vec{v}'\rangle, \tag{10}$$

and take into account the third term in (9) parametrically. This term is interpreted as the mass density flux due to diffusion. Using a well-known and very simple gradient hypothesis, first suggested in [11] (see also [18]) one sets the diffusion flux to be proportional to the gradient of the mass density. Let $\hat{\rho}$, unlike $\bar{\rho}$, denote the mass density which corresponds to the mean velocity field while the second term in (10) is taken into account. The quantity $\rho' \equiv \rho - \hat{\rho}$ denotes deviation of the mass density ρ from the value $\hat{\rho}$. Then

$$\langle \rho \vec{v}' \rangle = \langle (\hat{\rho} + \rho')\vec{v}' \rangle = \langle \rho' \vec{v}' \rangle, \tag{11}$$

and the mass density flux is as follows:

$$\hat{\rho}\vec{\bar{v}} + \langle \rho' \vec{v}' \rangle.$$

The first term describes redistribution of the mass density field via the mean current just as in the standard case. The second term reflects the contribution of pulsations. Next, in accordance with the gradient hypothesis we assume by definition

$$\langle \rho' \vec{v}' \rangle \equiv -k\nabla\hat{\rho}. \tag{12}$$

Here, the quantity k is a diffusion coefficient. Thus, the mass density flux is given by the expression $\hat{\rho}\vec{\bar{v}} - k\nabla\hat{\rho}$, which is in agreement with the above-made considerations concerning its possible form. Averaging of the continuity equation (6) leads to the mass density balance equation

$$0 = \langle \partial_t \rho + \mathrm{div}\rho\vec{v} \rangle = \partial_t \hat{\rho} + \mathrm{div}\hat{\rho}\vec{\bar{v}} - \mathrm{div}(k\nabla\hat{\rho}). \tag{13}$$

In the case $k = \mathrm{const}$ it is written as follows:

$$0 = \langle \partial_t \rho + \mathrm{div}\rho\vec{v} \rangle = \partial_t \hat{\rho} + \mathrm{div}\hat{\rho}\vec{\bar{v}} - k\Delta\hat{\rho} \tag{14}$$

or

$$d_t \hat{\rho} + \hat{\rho}\,\mathrm{div}\vec{\bar{v}} = k\Delta\hat{\rho}. \tag{15}$$

Equation (13) and its particular cases (14) and (15) are called the *mass density diffusion equation* of the moving continuum.
The fluid model based on the Navier–Stokes or Reynolds equations describes a non-divergent current. Any of these equations is obtained from more general one via zeroing of the divergence of velocity (or mean velocity, correspondingly). In this case, the mass density diffusion equation (15) simplifies to

$$d_t \hat{\rho} = k\Delta\hat{\rho}. \tag{16}$$

6. Concluding remarks

Let us summarize this research.

- In connection with repeated attempts to obtain the generalized continuity equation a question on rationality of such generalizations has been posed. The relevance of such attempts has been shown using a simple example which leads to a non-solvable problem.
- Close examination of the derivational history of the fluid mechanics equations allows one to maintain that the continuity equation, however it may be written, must contain terms dependent on the mass density and velocity only.
- The averaging procedure suggested in [12] has been considered. By being applied to the continuity equation it allows the derivation of the mass density diffusion equation of required form.

Some concluding remarks may be found in what follows. In particular, a modification of the mass density jump problem is considered.

(1) The above-discussed averaging procedure looks as if mass density is treated as an independent random quantity which might be written as $\bar{\rho} + \rho'$ where ρ' is a pulsation independent of $\vec{v}$. Despite the fact that the result would be the same this is only a formal resemblance. There exists a basic distinction from the standard averaging. The point is that if the above-stated considerations are not taken into account and ρ is considered to be an independent random quantity which undergoes averaging, then it should be such in each equation of the model. This is done in [18], for instance.

In contrast, within the procedure we suggested the mass density is not considered as a random quantity. It is calculated using (13) with respect to the mean velocity field and must not be averaged in all other equations of the model. In other words, all other equations of the model should contain $\hat{\rho}$ as the mass density. This also concerns all other variables of the problem, e.g. the internal energy or temperature (see the discussion of this topic in [12]). The velocity field is assumed to be the only quantity which contains a random component.

(2) A simple problem discussed in section 1, being modified using the generalized continuity equation, allows description of the initial profile smoothing as it follows from observations. Indeed, the system (2) now reads

$$\begin{aligned}
\partial_t \rho &= k\partial_{zz}\rho, \\
0 &= \partial_z p - \rho g, \\
\partial_t \theta &= \frac{1}{\rho C_\rho}\partial_t p + \kappa\partial_{zz}\theta, \\
\rho &= \rho(p,\theta),
\end{aligned} \tag{17}$$

where κ is a coefficient of thermal diffusivity. In addition to 3 boundary conditions must contain two conditions for ρ more. They may be obtained using the equation of state. Now, the mass density diffusion equation is of the same type as the heat equation. In the case of stationary pressure and a linear equation of state the first equation of the system (17) coincides with its third equation and models the required smoothing of initial distribution of mass density via heat conduction.

(3) In the oceanography, the mass density diffusion equation is sometimes used together with the continuity equation and the Navier–Stokes (Reynolds) equation. Actually, it closes the system instead of the equation of state. Seemingly, it was put in use by Lineikin, as done in his book [10] devoted to the theory of ocean currents.

(a) First, the equations of salt diffusion and the heat equation are written as

$$\begin{aligned}
d_t s &= k\Delta s, \\
d_t \theta &= k\Delta\theta.
\end{aligned} \tag{18}$$

Here s is the salinity and k is a coefficient of diffusivity.

(b) Next, based on the analysis of a particular equation of state of the sea water $\rho = \rho(p, s, \theta)$, one neglects compressibility as well as variability of quantities $\partial_s\rho$ and $\partial_\theta\rho$ and writes $\rho = \rho(s,\theta)$, $\partial_s\rho \equiv \alpha = \text{const}$, $\partial_\theta\rho \equiv \beta = \text{const}$. Thus, a linear equation of state is considered. The latter simplification allows writing derivatives of mass density in the form

$$\partial_a\rho = \alpha\partial_a s + \beta\partial_a\theta, \qquad a \in \{t, x, y, z\}. \tag{19}$$

(c) Finally, a weighted sum of equations (18) with weights α, β and expression (19) being taken into account gives the equation

$$d_t\rho = k\Delta\rho, \tag{20}$$

which is called the equation of the mass density diffusion.

This equation is used jointly with the equation of motion and the continuity equation. In Cartesian coordinates they are as follows:

$$\begin{aligned} d_t u &= -\frac{1}{\rho}\partial_x p + \nu\Delta u, \\ d_t v &= -\frac{1}{\rho}\partial_y p + \nu\Delta v, \\ 0 &= -\frac{1}{\rho}\partial_z p + g, \\ d_t\rho &+ \rho\,\mathrm{div}\vec{v} = 0. \end{aligned} \tag{21}$$

Here, standard notation for operators and variables is used and equations are cited with some insignificant simplifications. Other authors who use the mass density diffusion equation exploit similar systems of equations.

It should be noted that all such systems contain contradictions. First, comparing equations (20) and (21) it may easily be noted that Lineikin's model gives

$$d_t\rho = -\rho\,\mathrm{div}\vec{v} = k\Delta\rho,$$

which means compressibility of the fluid, although the opposite statement has been made. Moreover, the mass density diffusion equation itself points out this feature. Indeed, according to definition (see, e.g., [24, 25]) a continuum is called incompressible in the case where its every individual volume does not change in time or

$$d_t\rho = 0. \tag{22}$$

Second, the equations of motion (Navier–Stokes or Reynolds) which are used in geophysical problems and in Lineikin's model, in particular, are obtained for incompressible media (see, e.g., [18, 20]). Using these equations jointly with the continuity equation for compressible fluid actually means a non-standard definition of the viscous stress tensor (which appears to be proportional to velocity gradient rather than its symmetrical component), although neither Lineikin nor other authors seemingly had any such intention.

Thus, in all such cases the problems solved do not coincide with the declared ones. Surely, this does not mark the approximations used as bad. Simply, they differ from explicitly indicated at the formulation of the problem and most likely require another validation.

Is it necessary to exclude the mass density diffusion equation from consideration due to above-mentioned contradictions? From our viewpoint, just the reverse. It should be included into the system of equations of viscous/turbulent fluid, on different grounds, however. First, the mechanism of diffusion (of the mass density, particularly) seems to be rather general and is not strictly connected with linearity of the equation of state as occurs in Lineikin's derivation. Second, since the diffusion is one of the possible transfer mechanisms it seems reasonable to consider both mechanisms (convective and diffusive) within one and the same equation, as it is done in all other hydromechanical evolution equations. In addition, such diffusion fluxes naturally appear in conservation equations for mixture components (see, e.g., [24, 26]).

Thus, in our opinion, the mass density diffusion equation must be considered as a generalization of the continuity equation in accordance with the other authors cited above. Furthermore, the derivation of the mass density diffusion equation should not be based on linearity of the equation of state, since this produces a false idea that the diffusion mechanism of the mass density transfer is not general. We have tried to prove the above-made conclusion and to suggest a new derivation of this equation.

(4) Simplifications connected with application of the mass density diffusion equation to nearly incompressible fluids are not as evident as in the case of the standard continuity equation. The point is that incompressibility (22) now does not mean non-divergence of the velocity field. If a continuum model is based on the Navier–Stokes or Reynolds equations, it describes a non-divergent current with the mass balance equation of the form (15), i.e. a current of compressible fluid.

(5) The way the term $\langle \rho' \vec{v}' \rangle$ was described in (12) is rather simple and widely used but not unique, however (see, e.g., [18]).

(6) The suggested derivation of the mass density diffusion equation allows natural generalization onto the 4D case of the causal fluid model (see [17]). It is easy to show that such general equation is as follows:

$$d_t \hat{\rho} + \hat{\rho}\,\mathrm{div}\vec{\vec{v}}_4 = k\Box\hat{\rho},$$

where $\vec{\vec{v}}_4$ makes sense of 4-vector of mean velocity, and $\Box$ is d'Alembertian.

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